Arrow Polynomial and mod P Alexander Numbering

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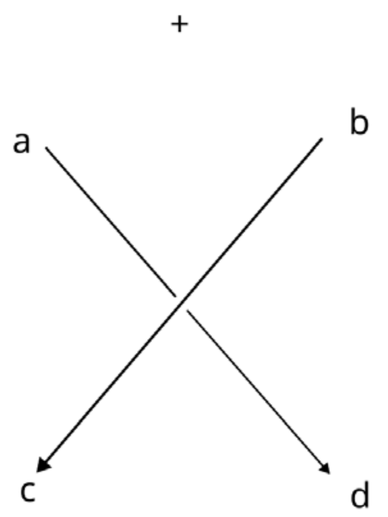
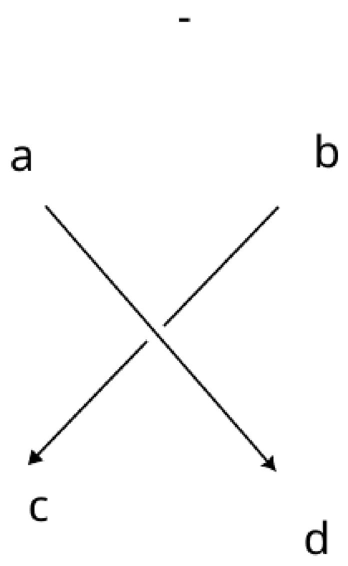
Alexander Numbering

Definition: An Alexander Numbering of a virtual link diagram L is a labeling of each semi-arch of L with a number $v \in Z$ such that at each crossing a=c and a+1 = b = d

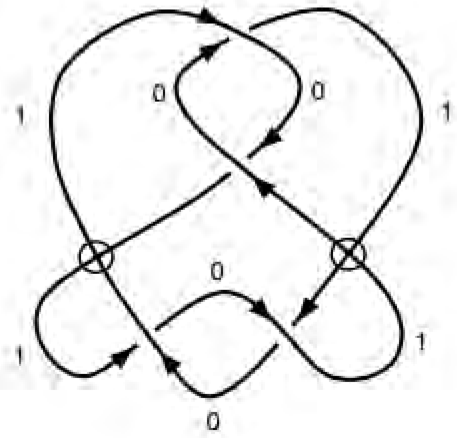
Definition: A knot is called almost classical if it admits an Alexander Numbering

Remark: All classical knots are almost classical





Alexander Numbering



Mod P Alexander Numbering

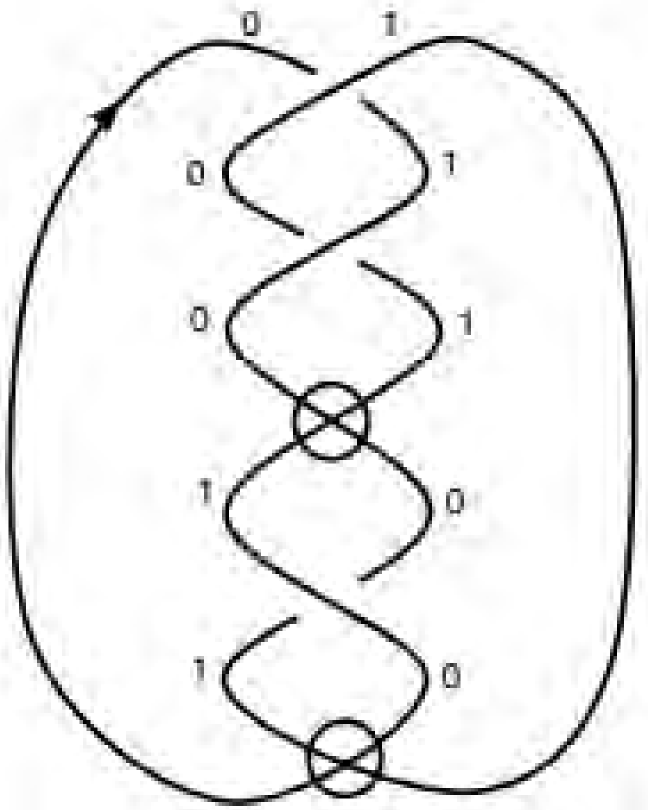
Definition: A mod p Alexander Numbering of a virtual link diagram L is a labeling of each semi-arch of L with a number $v \in Z_p$ such that at each crossing a=c and a+1 = b = d

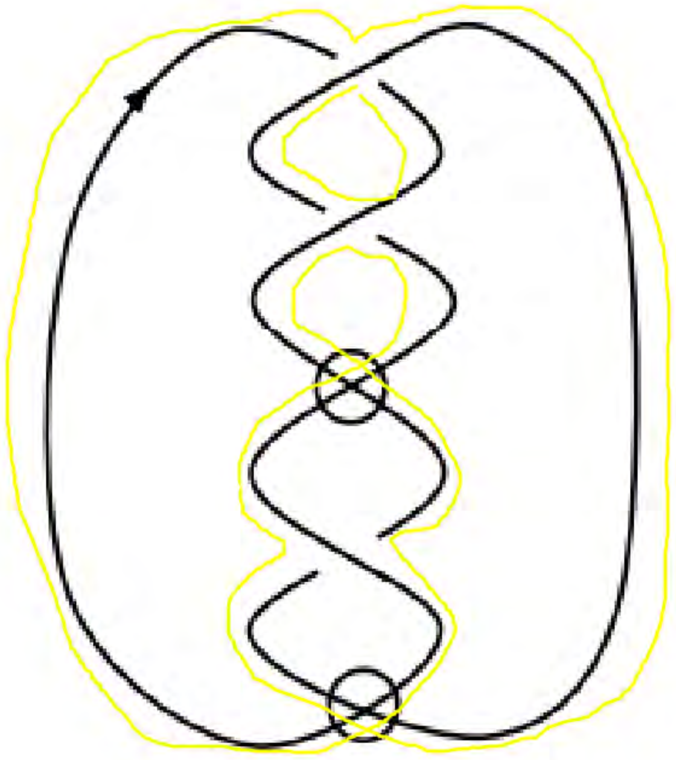
Definition 2: A knot is called called mod p almost classical if it admits a mod p Alexander Numbering.

Remark 1: All almost classical knots admit a mod 2 Alexander Numbering.

Remark 2: A virtual link is checkerboard colorable if and only if it is mod 2 almost classical.

Example of mod 2 Checkerboard Colorable Knot Which is Not Almost Classical





Arrow Polynomial and Checkerboard coloring

Theorem 4.3. *Let D be an oriented checkerboard colorable virtual link diagram. Then*

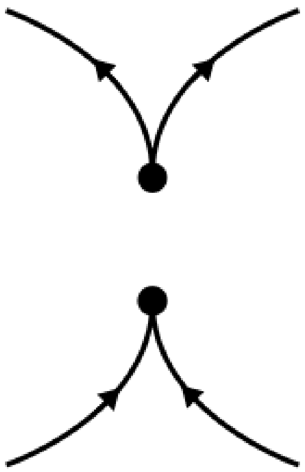
(1) $AS(D)$ only contains even integer; and

(2) for any summand $A^s K_{i_1}^{j_1} K_{i_2}^{j_2} \cdots K_{i_v}^{j_v}$ with $1 \leq i_1 < i_2 < \cdots < i_v$, $j_t \geq 1$ for $t = 1, 2, \cdots, v$, and $v \geq 1$ of $\langle D \rangle_{NA}$, we have $2i_v \leq \sum_{t=1}^{v} i_t \cdot j_t$. In particular, $\langle D \rangle_{NA}$ has no summands like $A^s K_i$.

Remark: Since any almost classical link admits a checkerboad coloring, this theorem holds for any almost classical link. This is not necessarily true for all mod p almost classical links.

In and Out Cusps

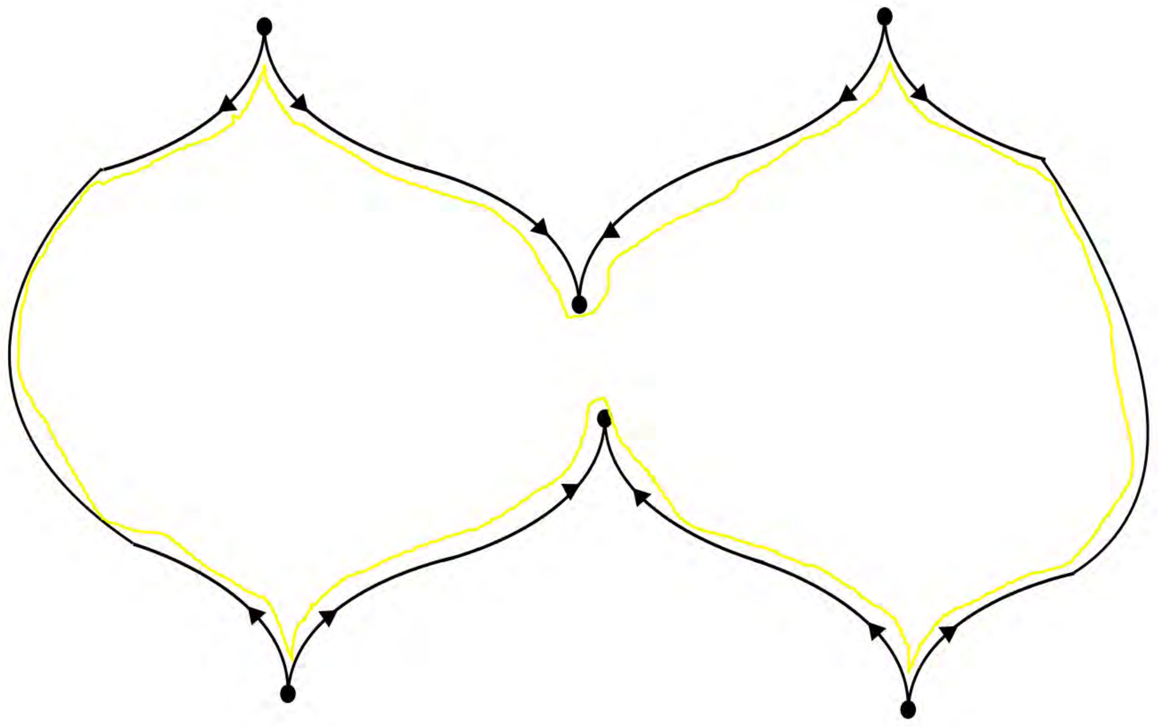
Out Cusp

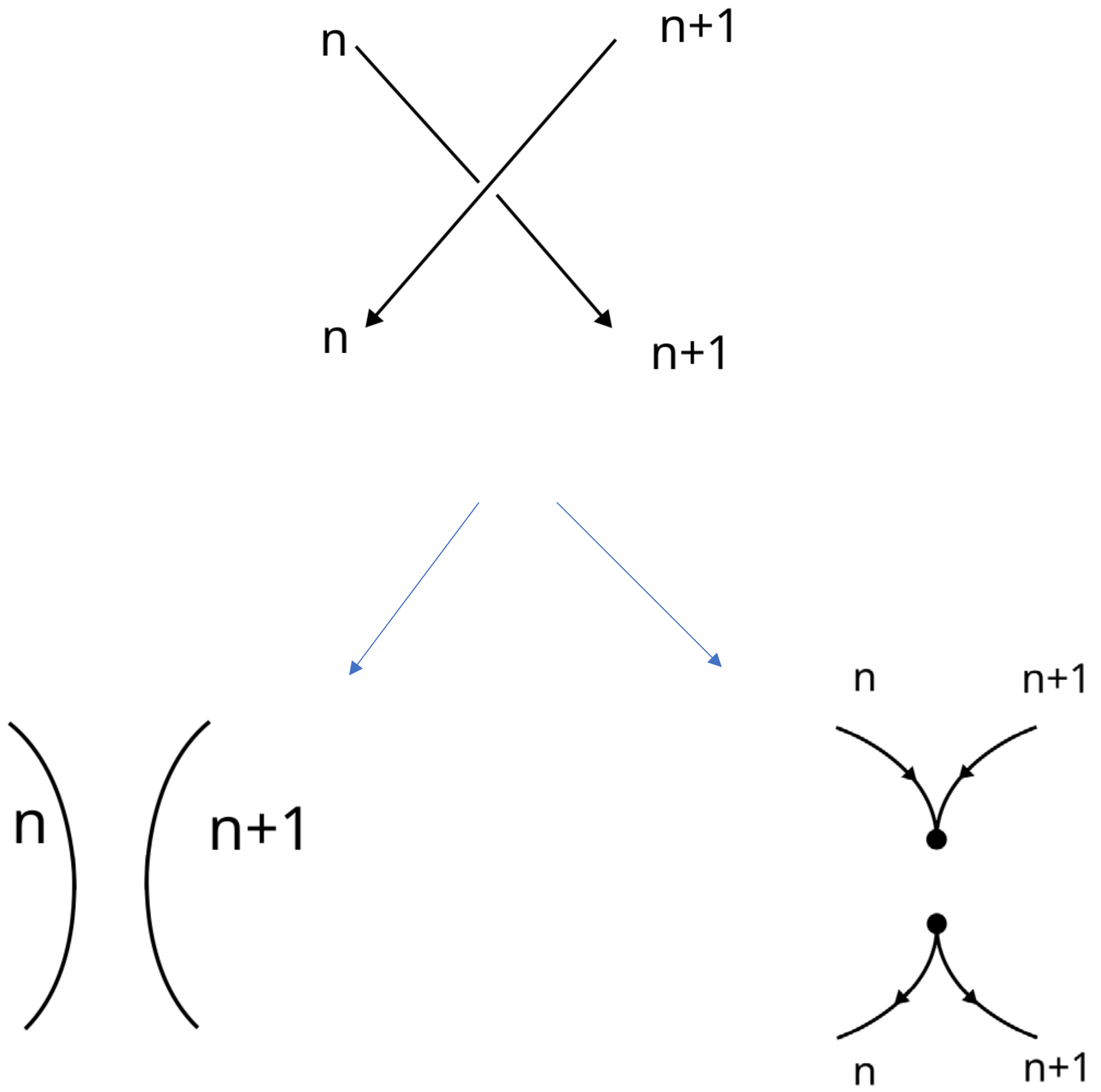


In Cusp

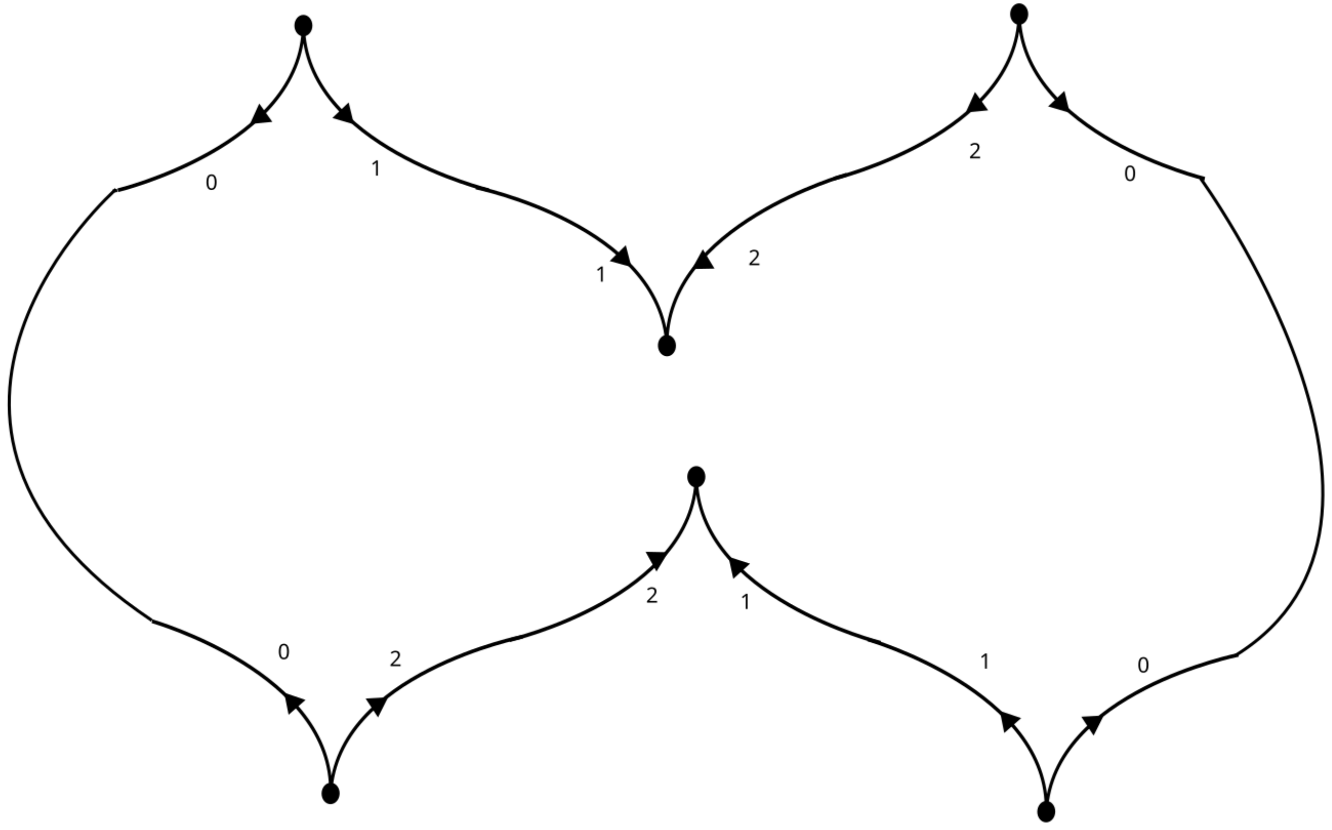
Word of a Circle Graph

$$0^{<}1^{>}2^{<}3^{>}4^{<}5^{>}$$



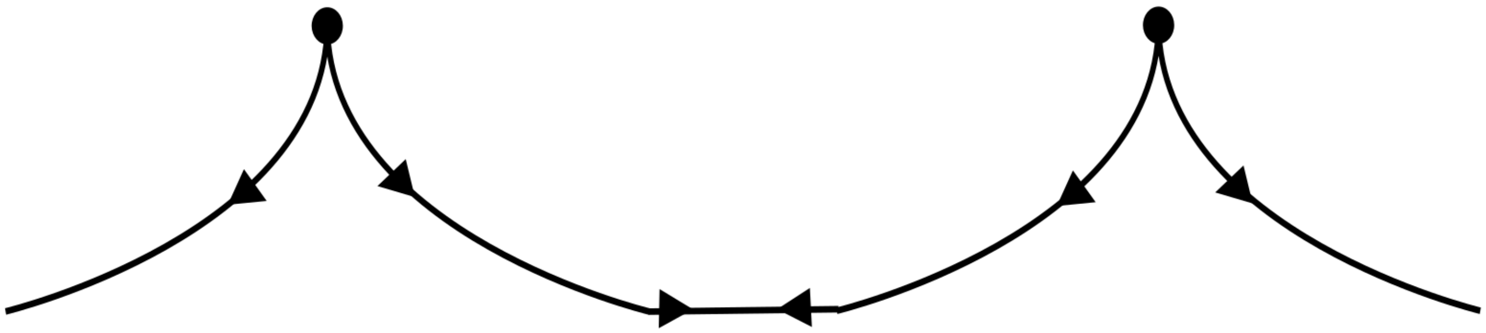

n
n+1
n
n+1
n
n+1
n
n+1
n
n+1

$$0^0 1^1 2^2 3^0 4^1 5^2$$



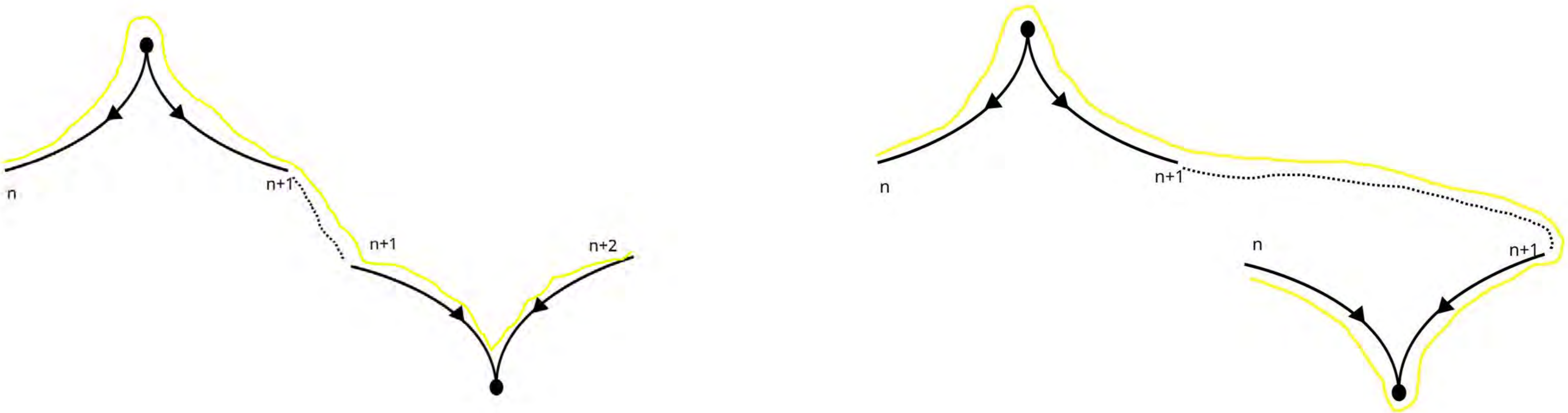
Theorem: For any odd integer p, let D be a mod p Alexander Numerable oriented virtual link diagram. Then any summand $A^s K_{i_0}^{j_0} K_{i_1}^{j_1} \ . \ . \ . \ K_{i_v}^{j_v}$ of $<D>_{NA}$, $p|i_t \ \forall 0 \leq t \leq v$

Claim 1: Any state circle contains in and out cusps alternativly. This result still holds if the state graph is reduced.



Definition: For any state circle σ we define an orientation that ignores cusps σ'. For any cusp i in the state circle σ' call i positively oriented if traversing σ' we move from the n to n+1 side of i as we traverse across i. Otherwise we call i negativly oriented.

Claim 2: Any adjacent cusps on a reduced state circle have the same orientation.



Proof: If any adjacent cusps have opposite orientation, they must be on the same side of the state circle. Therefore the state circle is not reduced.

Claim 3: Any reduced state graph contains only positively oriented or only negatively oriented cusps.

Proof: This follows directly from claim 2.

Claim 4: For any state circle with n cusps, n=2mp for some $m \in \mathbb{Z}_{\geq 0}$

Proof: Assume without loss of generality that the reduced circle graph σ contains positively oriented cusps. Let 0^n be any cusp in the word of σ. Since σ contains only positive cusps and since σ must contain an even number of cusps, the reduced word of σ must be $0^n 1^{n+1} \ . \ . \ . \ p^{n-1} \ . \ . \ . \ 2mp - 1^{n-1}$

A state circle with 2mp cusps contributes K_{mp} to the state. □

Corollary: Any almost classical virtual link has no cusps in any of its reduced state graphs.

Sources

- On arrow polynomials of checkerboard colorable virtual links
  by Qingying Deng , Xian'an Jin , Louis H. Kauffman
- Crowell's Derived Group and Twisted Polynomials by Daniel S. Silver
  Susan G. Williams